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Large gaps in one-dimensional cutting stock problems

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Abstract

Its linear relaxation is often solved instead of the one-dimensional cutting stock problem (1CSP). This causes a difference between the optimal objective function values of the original problem and its relaxation, called a gap. The size of this gap is considered in this paper with the aim to formulate principles for the construction of instances of the 1CSP with large gaps. These principles are complemented by examples for such instances.

Keywords: One-dimensional cutting stock problem; Linear relaxation; Approximate solution; Integer round up property

1. Introduction

The one-dimensional cutting stock problem (1CSP) can be formulated as follows: An unlimited number of identical stock material of length $w > 0$ is to be cut into b_i pieces of length $l_i > 0$ for $i \in I := \{1, \ldots, m\}$ such that a minimal amount of stock material is used. This problem has been investigated e.g. in [13,14]. Besides its obvious applications in cutting for instance shorter tubes with different lengths from longer ones with unit lengths, the problem has many other potential applications such as the storage of computer files on a minimum number of tapes. Moreover, it is a special case of the strip packing problem [5]. The cutting stock problem is equivalent to the bin packing problem, because every piece can be counted several times as a single item if it is demanded more than once. The problem is $\mathcal{NP}$-hard. Therefore approximation algorithms are used and approximate solutions are computed. Worst-case and average-case analysis of approximation algorithms became an important field of research in discrete optimization over the years [11]. As a result, a (relative) bound for the distance between the computed objective function value and the optimal one in the worst case, respectively in the average case is obtained. A newer overview on this topic is given in [2] for various online, semi-online and offline algorithms. Tight absolute bounds for the difference between the obtained and the optimal objective function values are still unproved besides trivial cases.

To check the quality of the obtained solution, it is helpful to know how tight the used bounds are. The aim of this paper is the construction of some instances of the 1CSP with a large difference (gap) between the optimal function values of the integer problem and its continuous relaxation. Focus is on the construction of instances with a small

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number of different piece lengths. We will construct instances with gaps of the values $\frac{13}{11}$ and $\frac{6}{5}$ using only 18 and 28 different pieces, respectively.

Several mathematical models for the 1CSP can be found in [1]. We will use the following model [4]:

$$\begin{aligned} &\sum_{j\in J} x_j \to \min, \\ &\sum_{j\in J} a^j x_j = b, \\ &x_j \in \mathbb{N}, \quad j \in J, \end{aligned} \tag{1.1}$$

where $b = (b_i)_{i\in I}$ and $J = \{1, \dots, n\}$ is the index set of (given or all) different cutting patterns $a^j = (a_{ij})_{i\in I} \in \mathbb{N}^m$. These patterns indicate the possibilities for cutting one stock material piece into shorter pieces. The number $a_{ij} \in \mathbb{N} := \{0, 1, 2, \dots\}$ determines how often piece i is cut out of one stock material piece in the jth pattern. Clearly, each such cutting pattern has to satisfy the condition

$$\sum_{i\in I} a_{ij} l_i \leqslant w. \tag{1.2}$$

Usually the number of different cutting patterns is very large. In the following we will abbreviate an instance of the 1CSP by the tuple (m, w, l, b) with $m \in \mathbb{N}$, $w \in \mathbb{R}_+$, $l \in \mathbb{R}^m_+$ and $b \in \mathbb{N}^m$.

One of the basic ideas in model (1.1) is that overproduction is useless and that unused parts of the stock material pieces are waste. Since a subset of cut pieces in one pattern can be used to construct a new pattern, the formulation of the model with equality constraints is without loss of generality. Moreover the problem has an optimal solution if $l_i \leqslant w$ for all $i \in I$ and $\{a^j : j \in J\}$ is sufficiently comprehensive.

The linear relaxation of (1.1) is

$$\begin{aligned} &\sum_{j\in J} x_j \to \min, \\ &\sum_{j\in J} a^j x_j = b, \\ &x_j \geqslant 0, \quad j \in J. \end{aligned} \tag{1.3}$$

Let z_{D} and z_{C} denote respectively the optimal objective function values of the cutting stock problem (1.1) and its linear relaxation (1.3).

To check the quality of the obtained solution, tight bounds for z_{D} are necessary. The *material bound*

$$z_{\mathrm{M}} := (l^\top b)/w$$

is in general too weak. The continuous relaxation is more suitable. Early a-posteriori investigations suggested the conjecture that

$$\Delta := z_{\mathrm{D}} - z_{\mathrm{C}}$$

is less than 1. This was disproved in the papers [7,8] where instances (i.e. examples) have been formulated for which the integer round up property (IRUP) does not hold, i.e. for which $z_{\mathrm{D}} > \lceil z_{\mathrm{C}} \rceil$ is true. Therefore, the detection of the correct value z_{D} is difficult. Moreover, it has been shown that it is $\mathcal{NP}$-hard to verify if an instance of (1.1) has IRUP.

An instance of (1.1) belongs to the so-called *divisible case*, if w/l_i is integer for all $i \in I$. In this case it has been proved in the papers [6,15] that $\Delta < 2$. More recently, $\Delta < \frac{7}{5}$ has been shown for all instances of the divisible case [12], especially $\Delta < \frac{5}{4}$ if $w/l_i \in \{1, 2, \dots, 18\}$ for all i.

Until now it remains an open problem, whether $\Delta < 2$ is valid for all instances of (1.1) or, more general, how large the supremum $\alpha(m)$ of the possible gap can be. Other questions that have been posed are whether instances with gap larger than 1 are rather seldom or can appear only for instances with large data. In this paper we will give partial answers to these questions. We give a new construction principle for instances with gaps essentially larger than 1 in Theorem 3.3 below. Applying this principle we derive well-defined instances with larger gaps for smaller instances. With other words, we intend to formulate new results on lower bounds for $\alpha(m)$.

2. History of large gaps

A first instance of the cutting stock problem violating IRUP can be found in [7,8]. Later on, instances with gaps of $\frac{31}{30}$ [3] and $\frac{137}{132}$ [15], both for instances with three different pieces have been found. Using instances with six and five different pieces, the gap reached $\frac{101}{96}$ and $\frac{16}{15}$, respectively [17]. Again using instances with five and six pieces, gaps of $\frac{19}{18}$ and $\frac{35}{33}$ have been reported in [10], respectively. Using different construction principles, new instances with even larger gaps have been found in the Ph.D. thesis [12]. Known gaps are

m	1	3	5	6	7	8	10	11	15	18	23	32
Δ	<1	$\frac{137}{132}$	$\frac{16}{15}$	$\frac{38}{35}$	$\frac{11}{10}$	$\frac{10}{9}$	$\frac{39}{35}$	$\frac{149}{132}$	$\frac{51}{44}$	$\frac{7}{6}$	$\frac{13}{11}$	$\frac{6}{5}$

It should be noted that besides certain specific instances with large gaps, classes of instances depending on certain parameters have also been found such that all members of these classes violate IRUP [9,14,12].

Example (*Rietz et al. [14]*). Let $p \in \mathbb{N}\backslash\{0\}$ and let w be the least common multiple of $3p, 3p+1, 9p+2$. Then, the instance

$$\left(3, w, \left(\frac{w}{3p}, \frac{w}{3p+1}, \frac{w}{9p+2}\right)^{\top}, (3p-1, 3p, 6)^{\top}\right)$$

has gap $\Delta > 1$.

3. Construction principles

In constructing instances with large gaps it is often necessary to perturb the lengths of the pieces such that certain cutting patterns become infeasible. Having this in mind the following results can be helpful. To abbreviate cutting patterns we use the unit vectors $e^i \in \mathbb{R}^m$ with $e^i_j \in \{0, 1\}$ for all $j \in I$ and $e^i_j = 1$ iff $i = j$. A cutting pattern $e^i + 3e^j$ means that piece i is cut once, piece j is cut three times and that no other pieces are cut in that pattern.

Lemma 3.1. *Consider the instance* $(9, w, l, (1, 1, \ldots, 1)^{\top})$ *with* $z_D > 3$. *Let the following cutting patterns be feasible, i.e. satisfy inequality* (1.2):

$$a^1 = e^1 + 2e^4, \quad a^2 = e^2 + 2e^5, \quad a^3 = e^3 + 2e^6,$$
$$a^4 = e^1 + e^8 + e^9, \quad a^5 = e^2 + e^7 + e^9, \quad a^6 = e^3 + e^7 + e^8.$$

Then, we have the following implications for the lengths of the pieces:

$$l_1 \neq l_2, \quad l_4 \neq l_5, \quad l_5 \neq l_7, \quad l_6 \neq l_7, \quad l_7 \neq l_8.$$

If $l_4 = l_7$ *holds too, then*

$$l_5 \neq l_8, \quad l_6 \neq l_9, \quad l_1 + l_5 + l_6 > w, \quad 2l_1 > l_2 + l_3 \quad \textit{and} \quad z_M < 3.$$

Proof. We show the result indirectly by constructing certain contradictions. Feasibility of the cutting patterns a^1–a^6 implies $z_C \leqslant 3$ (since $x_j = 0.5$ for all j is a feasible solution of problem (1.3)) but $z_D \geqslant 4$ by assumption. Hence, IRUP is violated.

- Assume $l_1 = l_2$. Then, $l_1 + l_4 + l_5 \leqslant w$ and $l_2 + l_4 + l_5 \leqslant w$ by $l^{\top}(a^1 + a^2) \leqslant 2w$. Due to $z_D > 3$ and feasibility of the patterns a^4, a^5 this implies $l_3 + l_6 + l_7 > w$ and $l_3 + l_6 + l_8 > w$. Hence, we get $2w < (l_3 + l_6 + l_7) + (l_3 + l_6 + l_8) = (l_3 + 2l_6) + (l_3 + l_7 + l_8) \leqslant 2w$ by feasibility of a^3 and a^6. This contradiction implies that $l_1 \neq l_2$.
- The same arguments can be used if $l_4 = l_5$ is assumed since feasibility of a^1 and a^2 again leads to $l_1 + l_4 + l_5 \leqslant w$ and $l_2 + l_4 + l_5 \leqslant w$.

- Assume now that $l_5 = l_7$. Then, by $l_2 + l_5 + l_7 \leqslant w$, $l_1 + l_8 + l_9 \leqslant w$ and $z_D > 3$ we derive $l_3 + l_4 + l_6 > w$. Analogously, $l_3 + l_5 + l_8 \leqslant w$ and $l_2 + l_7 + l_9 \leqslant w$ imply $l_1 + l_4 + l_6 > w$. This together has the consequence $2w < (l_3 + l_4 + l_6) + (l_1 + l_4 + l_6) = (l_1 + 2l_4) + (l_3 + 2l_6) \leqslant 2w$ due to feasibility of the cutting patterns a^1 and a^3. This contradiction shows $l_5 \neq l_7$.
- In a similar fashion, the assumption $\ell_6 = \ell_7$ can be used to get the contradiction: $l_3 + l_6 + l_7 \leqslant w$ and $l_1 + l_8 + l_9 \leqslant w$ imply $l_2 + l_4 + l_5 > w$; $l_2 + l_6 + l_9 \leqslant w$ and $l_3 + l_7 + l_8 \leqslant w$ lead to $l_1 + l_4 + l_5 > w$. Combining both inequalities with feasibility of a^1 and a^2 results in the desired contradiction.
- Assume $l_7 = l_8$. Then, feasibility of a^3 and a^6 implies $l_3 + l_6 + l_7 \leqslant w$. Since a^4 is feasible we obtain $l_2 + l_4 + l_5 > w$ by $z_D > 3$. Analogously, from $z_D > 3$ and $l_3 + l_6 + l_8 \leqslant w$ combined with feasibility of a^5, it follows that $l_1 + l_4 + l_5 > w$. Adding both derived inequalities we obtain $2w < (l_1 + 2l_4) + (l_2 + 2l_5) \leqslant 2w$ which contradicts feasibility of the patterns a^1 and a^2.
- Now let $l_4 = l_7$ be valid.
  - Inequality $l_1 + l_5 + l_6 \leqslant w$ would imply a contradiction to $z_D > 3$ due to feasibility of a^5 and a^6. Hence, $l_1 + l_5 + l_6 > w$ implying $2l_1 > 2w - 2l_5 - 2l_6 \geqslant l_2 + l_3$ by the assumption for the patterns a^2 and a^3.
  - Assume $l_5 = l_8$. Feasibility of a^6 implies $l_3 + l_4 + l_5 \leqslant w$ and thus $l_2 + l_6 + l_7 > w$ by $z_D > 3$ as well as $l_1 + l_8 + l_9 \leqslant w$. Comparing this with $l_2 + l_7 + l_9 \leqslant w$ we get $l_6 > l_9$. Hence, $l_3 + l_6 + l_9 < w$ by feasibility of a^3, but then, feasibility of $e^1 + e^4 + e^7$ (see a^1) and $e^2 + e^5 + e^8$ (cf. a^2) would imply $z_D \leqslant 3$ which contradicts our assumption.
  - Interchanging the indices the same idea can be used to treat the case $l_6 = l_9$.
  - By $z_M \leqslant z_C$ we derive $z_M \leqslant 3$. Assume $z_M = 3$. Then, all the patterns a^1–a^6 are without trimloss. This especially leads to $l_2 + l_7 + l_9 = w = l_3 + l_4 + l_8$. Hence, $z_M = 3$ implies $l_1 + l_5 + l_6 = w$ too, leading to $z_D = 3$. This contradiction concludes the proof. □

Using the assertions in the last lemma, a first instance violating IRUP is obtained in the following theorem.

Theorem 3.2. *Let $p, q \in \mathbb{N}$, $p \leqslant q$ be arbitrarily chosen and consider the instance $E_0(p, q)$ of the 1CSP*

$$(8, 33 + 3p + q, (21 + p + q, 19 + p + q, 15 + p + q, 10 + p, 9 + p, 7 + p, 6 + p, 4 + p)^\top, e + e^6)$$

with $e = (1, 1, \ldots, 1)^\top$. Then, $z_C \leqslant 3$ and $z_D = 4$.

Proof. Denote this instance by $E_0(p, q)$. After reordering the pieces in $E_0(p, q)$ and replacing the sixth piece by two new pieces each of which is needed one time, we transform $E_0(p, q)$ into

$$(9, 33 + 3p + q, (19 + p + q, 15 + p + q, 21 + p + q, 7 + p, 9 + p, 6 + p, 7 + p, 4 + p, 10 + p)^\top, e).$$

For this instance all the cutting patterns a^1–a^6 in Lemma 3.1 are feasible. Hence, $z_C \leqslant 3$.

Assume that $z_D = 3$. Then, since $3w - l^\top b = 1$, exactly one of the patterns used in an optimal solution of problem (1.1) has waste 1. No two of the first three pieces fit into one pattern since

$$w - (19 + p + q) - (15 + p + q) = p - q - 1 \leqslant -1.$$

Because of $w - (21 + p + q) - (10 + p) = 2 + p > 1$ no cutting pattern containing two pieces can be used in an optimal solution of problem (1.1). Hence, by $e^\top b = 9$, the assumption $z_D = 3$ implies that all used patterns contain exactly three pieces. The only three-piece pattern containing piece number 1 is $e^1 + e^6 + e^8$ since

$$w - (21 + p + q) - (7 + p) - (4 + p) = 1$$

and this pattern has waste 1. This implies that it is not possible to cut piece number 2 without waste. Therefore, it is not possible to use only three cutting patterns in an optimal solution of (1.1) and the assumption $z_D = 3$ was wrong. □

Now, it is our aim to compose the instance in Theorem 3.2 with other instances thus getting examples with large gaps. This approach has been successfully exploited in [13,12].

Let $E_1 = (m_1, w_1, l^1, b^1)$ and $E_2 = (m_2, w_2, l^2, b^2)$ denote two instances of (1.1) with $w_1 = w_2$. The *composed instance* $E := E_1 \oplus E_2$ of (1.1) consists of the task to cut all the $m_1 + m_2$ pieces of the lengths in both the vectors l^1

and l^2 according to the demands in both the vectors b^1 and b^2, respectively. The same idea of composing two instances into one instance is also possible if the lengths of their stock materials are different. For doing so multiply all the lengths of both the stock material and the pieces to cut from it in one (or both) instance by one common multiplier to adjust the stock material lengths of both instances. Multiplication of all those lengths by a positive number gives an equivalent instance with the same set of feasible cutting patterns and the same optimal solutions in both (1.1) and (1.3). As example consider the instances (1, 2, 1, 1) and (1, 5, 2, 2) which can be composed to the new instance $(1, 2, 1, 1) \oplus (1, 5, 2, 2)$ which is equivalent to $(2, 10, (5, 4)^\top, (1, 2)^\top)$.

Theorem 3.3. *Consider an instance $E = (m, w, l, b)$ of problem* (1.1) *with the following properties*: $l_1 > l_2 > \cdots > l_{m-1} > 2l_m$ *and* $l_m \leqslant w/4$. *Moreover, assume that this instance is sensitive w.r.t.* b_m, *i.e. assume that its optimal function value increases if* b_m *is increased by* 1. *Then, there are numbers* $p, q \in \mathbb{N}$ *such that the instance* $E' = E \oplus E_0(p, q)$ *has gap* $\Delta(E') = 1 + \Delta(E)$.

Proof. Obviously, the length l_m can be decreased to some $0 < \lambda < l_m$ without producing new cutting patterns. The resulting instance is equivalent to E. Let $s := l_m/w,\ \ r := \lambda/w < \frac{1}{4}$.

1. First, it is shown that there exist $p, q \in \mathbb{N}$ with

$$p \leqslant q \wedge 10 + p \leqslant s * (33 + 3p + q) \wedge 4 + p > r * (33 + 3p + q). \tag{3.1}$$

The last two inequalities are equivalent to

$$\frac{10 + p}{s} - 33 - 3p \leqslant q < \frac{4 + p}{r} - 33 - 3p. \tag{3.2}$$

Due to $\frac{4+p}{r} - \frac{10+p}{s} = \frac{p*(s-r)+4s-10r}{r*s}$ and $r < s$ there is some $q \in \mathbb{N}$ satisfying (3.2) for sufficiently large p. Since $r < \frac{1}{4}$, the limit of $\frac{4+p}{rp} - \frac{33}{p} - 3$ for $p \to \infty$ is $\frac{1}{r} - 3 > 1$, and it is thus possible to guarantee $p \leqslant q$ for sufficiently large p, too.

2. Second, investigate the instance $E' = E \oplus E_0(p, q)$. To start, adjust the instance E such that $w = 33 + 3p + q$ by multiplication of all the lengths in E by a suitable positive constant. Then, (3.1) implies $l_m \geqslant 10 + p$ and, due to $l_{m-1} > 2l_m$, we derive

$$15 + p + q + l_{m-1} > 15 + p + q + 2l_m \geqslant 15 + p + q + 2(10 + p) > 33 + 3p + q.$$

Hence, the three largest pieces in $E_0(p, q)$ cannot be combined with any of the pieces with length $l_1, \ldots, l_{m-1}$. Possibly there are cutting patterns combining one of the three largest pieces in $E_0(p, q)$ with the shortest piece in E but, since $sw = l_m \geqslant 10 + p$, this piece can be substituted by the piece of length $10 + p$ of E_0. Thus, no essentially new cutting pattern arises containing one of the three largest pieces in E_0. Deleting the shortest piece of length $4 + p$ from E_0 leads to $z_D(E_0) = 3$ but adding it to E increases $z_D(E)$ by 1 due to $4 + p \geqslant \lambda$ and the assumption of the theorem. The same is true if a certain number of pieces from E is moved to E_0 and is put into the fourth cutting pattern (the one without one of the three largest pieces in E_0). To see this consider all the pieces of E_0 in this fourth cutting pattern as belonging to E. Then, the rest of the pieces in E_0 can be cut with three patterns, but to cut all the other pieces we need at least $z_D(E) + 1$ cutting pattern by the assumption of the theorem that $z_D(E)$ increases if b_m is increased and the shortest piece in E_0 is longer than λ. This shows that $z_D(E') = z_D(E_0) + z_D(E)$.

3. By $z_C(E') \leqslant z_C(E_0) + z_C(E)$ we get $\Delta(E') \geqslant \Delta(E) + 1$. Instance E_0 contains three pieces of type 1 (these are pieces with a length larger than $w/2$). Hence, deleting one of the shorter pieces and moving it to E cannot have any impact on $z_C(E_0) = 3$. Vice versa, adding any piece from E to E_0 will imply an increase of z_C since the material bound of the resulting instance is then larger than 3. Last but not least, an exchange of the shortest piece in E with one of the shorter pieces in E_0 can also not lead to a decrease of z_C by equivalence. Hence, $z_C(E') = z_C(E_0) + z_C(E)$ is verified. Due to Theorem 3.2 we have $z_D(E_0) = 4$. The assumption $l_m \leqslant w/4$ implies $p + 10 \leqslant l_m \leqslant w/4$ and thus $w - (15 + p + q) = 18 + 2p < w/2$. Hence, piece 3 of E_0 with length $15 + p + q$ is of type 1 [12] and cannot be combined with one of the larger pieces. This implies $z_C = 3$ proving the theorem. □

Remark 3.4. It should be noted that $b_m = 0$ is one possible selection in Theorem 3.3. This means that the maximal possible trimloss in a cutting pattern used in an optimal solution is smaller than one half of the length of the shortest piece. This demand in Theorem 3.3 is most difficult to be satisfied.

Using Theorem 3.3 the following instances with large gaps have been constructed. In all these instances the relations $z_{\mathrm{M}} < z_{\mathrm{C}}$ and $z_{\mathrm{D}} = \lceil z_{\mathrm{C}} \rceil + 1$ are valid.

$$E_1 := (10, 924, l^1, (1, 1, 1, 2, 3, 6, 1, 2, 1, 1)^\top),$$
$$l^1 = (764, 762, 758, 308, 231, 84, 83, 81, 80, 78)^\top,$$
$$z_{\mathrm{C}} = 5 - \tfrac{17}{132}, \quad \Delta = \tfrac{149}{132}.$$
$$E_2 := (14, 1320, l^2, (1, 1, 1, 1, 1, 2, 1, 1, 3, 2, 1, 2, 1, 1)^\top),$$
$$l^2 = (1088, 1086, 1082, 444, 443, 440, 438, 434, 330, 120, 119, 117, 116, 114)^\top,$$
$$z_{\mathrm{C}} = 6 - \tfrac{7}{44}, \quad \Delta = \tfrac{51}{44}.$$
$$E_3 := (16, 1260, l^3, (1, 1, 1, 1, 1, 1, 1, 2, 1, 2, 1, 3, 1, 2, 1, 1)^\top),$$
$$l^3 = (946, 944, 940, 630, 560, 532, 476, 406, 392, 364, 350, 161, 160, 158, 157, 155)^\top,$$
$$z_{\mathrm{C}} = 6 + \tfrac{5}{6}, \quad \Delta = \tfrac{7}{6}.$$
$$E_4 := (18, 1386, l^4, (1, 1, 1, 1, 1, 1, 1, 1, 2, 1, 1, 1, 1, 10, 1, 2, 1, 1)^\top),$$
$$l^4 = (1142, 1140, 1136, 694, 693, 690, 466, 465, 462, 460, 456, 348, 346, 126, 125, 123, 122, 120)^\top,$$
$$z_{\mathrm{C}} = \tfrac{86}{11}, \quad \Delta = \tfrac{13}{11}.$$
$$E_5 := (28, 6810, l^5, b^5),$$
$$b^5 = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 1, 1, 1, 1, 2, 1, 1, 5, 1, 2, 1, 3, 1, 2, 1, 1)^\top,$$
$$l^5 = 5516, 5514, 5510, 4166, 4146, 4106, 3548, 3546, 3542, 2274, 2273, 2270,$$
$$2268, 2264, 1635, 1634, 1632, 1631, 1629, 1362, 1352, 1332, 1322, 651, 650, 648, 647, 645)^\top,$$
$$z_{\mathrm{C}} = \tfrac{59}{5}, \quad \Delta = \tfrac{6}{5}.$$

Remark 3.5. These instances especially show that the gaps presented in the tabular in Section 2 can be obtained even with instances using a smaller number of different pieces.

For a correct verification of the optimal function values z_{D} of these instances a cutting plane algorithm [16] can be used. Some comments should explain how to find appropriate instances for Theorem 3.3.

The pieces 1–3 and 6–10 of the first instance E_1 are the pieces in the instance E_0 with the selection $p = 74$ and $q = 669$ as parameters. The pieces 4–6 originally formed the instance

$$E_{\mathrm{ST}} = (3, 132, (44, 33, 12)^\top, (2, 3, 6)^\top)$$

with gap $\Delta(E_{\mathrm{ST}}) = \frac{137}{132}$ in [15]. This instance does not satisfy the assumptions of Theorem 3.3 since increasing b_m by 1 does not imply that $z_{\mathrm{D}}(E_{\mathrm{ST}})$ increases by 1, too. Since E_{ST} belongs to the divisible case [15] we have $z_{\mathrm{M}}(E_{\mathrm{ST}}) = z_{\mathrm{C}}(E_{\mathrm{ST}})$. Therefore, $z_{\mathrm{M}}(E_{\mathrm{ST}}) = z_{\mathrm{C}}(E_{\mathrm{ST}})$ decreases to $2 - \frac{17}{132}$ if one piece of length 12 is deleted. With $z_{\mathrm{D}} = 2$ for the resulting instance and using the result for the original instance we see that the resulting instance satisfies the assumptions in Theorem 3.3. Now, using the assertion of Theorem 3.3 instance E_1 arises.

In both the instances E_2 and E_4 the starting point was E_{ST}, but here some of the pieces of length $w/11$ were substituted by several pieces forming a non-IRUP instance, e.g.

$$(5, 75 + 3t, (29 + t, 28 + t, 25 + t, 23 + t, 19 + t)^\top, (1, 1, 2, 1, 1)^\top), \quad t > -17. \tag{3.3}$$

In this last instance all piece lengths tend to $w/3$ for $t \to \infty$.

The construction of the instance E_5 was much more complicated. Originally the pieces 4–24, forming the needed instance for Theorem 3.3, are composed from several non-IRUP instances, namely instance (3.3), an instance $E_0(p, q)$

with $10 + p = w/5$, another instance $E_0(p', q')$ with $10 + p' \approx w * \frac{11}{45}$ and four pieces of length $w/5$. If pieces with nearly the same length are counted as equal pieces, a model instance $(5, 45, (27, 23, 15, 11, 9)^\top, (3, 3, 6, 6, 10)^\top)$ is constructed which satisfies the condition that the trimless patterns (satisfying $45 = 3 * 15 = 27 + 2 * 9 = 23 + 2 * 11$) must not be used at least once. Then the maximum trimloss in a pattern used in an optimal integer solution of the model instance is 4, and this is less than half of the shortest piece. Before the final step of construction the piece of length 1302 is replaced by two pieces of length 651, such that Theorem 3.3 is still applicable.

4. Conclusion and open questions

A new construction principle for instances with a gap between the optimal objective functions values of the one-dimensional cutting stock problem and its linear relaxation being essentially larger than 1 has been obtained. Using this principle it is possible to increase the largest known gaps using small instances. Using this approach the table of known gaps is Section 2 can be replaced by

m	1	3	5	6	7	8	10	14	16	18	28
Δ	< 1	$\frac{137}{132}$	$\frac{16}{15}$	$\frac{38}{35}$	$\frac{11}{10}$	$\frac{10}{9}$	$\frac{149}{132}$	$\frac{51}{44}$	$\frac{7}{6}$	$\frac{13}{11}$	$\frac{6}{5}$

The entries in this table are far from the upper bound $\Delta < \max\{2, (m + 2)/4\}$, which was proved in [13,12]. The table yields the conjecture that the maximal possible gaps could be bounded by $1 + C * \ln m$ with a constant C. It is a first open question whether such a bound can be verified. A second unanswered question is whether the gap can be bounded by 2 or not. And last, but not least, the search for instances with larger gaps remains a challenging task. In our opinion this should be complemented by the verification of new construction principles for such instances. Probably a gap $\frac{6}{5}$ can be constructed with fewer different pieces, but this shall be postponed to a later work.

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